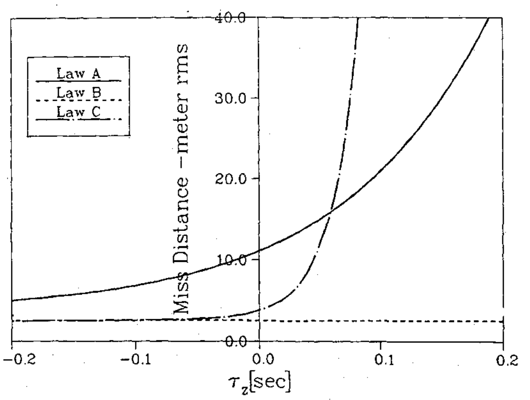

Fig. 3 Rms miss distance vs the zero placement for a third-order autopilot for various modern guidance laws.

The comparison is performed by computation of the root mean square (rms) miss due to target's random maneuver and glint by the adjoint method.[8] The simulations are performed against a step target's acceleration whose initiation instant is uniformly distributed over the flight time.[9] All the the results are for $g = \infty$ in Eq. (10).

Figure 2 presents curves of the effective navigation ratio $N' = (t_f - t)^2 \Lambda(t_f - t)$ vs time to go $t_{go} = t_f - t$ for the different guidance laws and for minimum and nonminimum phase autopilots, respectively. One can see that the effective navigation ratio goes to infinity for the minimum phase autopilot and to the negative infinity for the nonminimum phase autopilot.

Figure 3 shows the rms miss distance vs the autopilot's zero for the different guidance laws, respectively. The results are derived for a target acceleration of $5g$ uniformly distributed over 5 s and glint noise with spectral density of $1\,\mathrm{m}^2/\mathrm{Hz}$.[10] One can see that for minimum phase autopilot there is only a slight difference between the various guidance laws. However, for nonminimum phase autopilots, the superiority of the full-state feedback modern guidance law is well demonstrated.

One can see that the full-state feedback modern guidance law is quite insensitive to the zero location. It is explained as follows. For perfect knowledge of the states, the guidance law brings the miss to very small values. As the guidance law is fed by the estimates of the states, the miss is caused mostly by the estimation error. Hence, the miss is quite insensitive to the zero location.

Conclusions

The general form presented for an optimal guidance law enables one to systematically synthesize modern guidance laws for high-order autopilots. For a third-order nonminimum phase autopilot, such a guidance law gives improved performance with respect to a first-order guidance law.

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Optimal Nonlinear Compensator

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Introduction

OPTIMAL feedback regulation of analytic, but nonlinear, dynamic processes had been presented in Refs. 1 and 2. Those works only considered the deterministic case, where no dynamic noise is present in the process and the state is fully accessible.

In Ref. 3, an extended theory was developed that carries a similar methodology over to the fully stochastic situation. There, symmetric plants were considered, with their nonlinearities being cubic (e.g., saturations), including a distribution matrix, quadratic in the state, for the dynamic noise, as well as nonlinear and noisy measurements.

The resulting "compensator" is an optimal nonanticipative law that feeds back various moments of the state probability distribution, conditioned on the measurements. Consistently optimal algorithms were also given for on-line updating of those moments (i.e., an optimal nonlinear estimator). Those results were summarized by Shefer and Breakwell.[4]

In the present Note, we demonstrate the fully stochastic theory on a process with quadratic nonlinearities. Such nonlinearities can describe both dynamic couplings (e.g., gyroscopic and Coriolis terms) and static ones (the sensory type). They can adequately represent dissipative phenomena such as induced drag. Here, a scalar example has been selected to make the analysis more transparent. Higher order examples will be addressed in a future paper.

Outline

The problem in discrete form appears as follows. Given

State update:

$$x_{n+1} = x_n + \epsilon F_1 x_n^2 + u_n + w_n, \qquad E\{w_n w_m\} = 1\delta_{mn}$$

Measurement:

$$y_n = x_n + \epsilon H_1 x_n^2 + w_n', \qquad E\{w_n' w_m'\} = 2\delta_{mn}$$

ϵ is a small parameter and w and w' are Gaussian noises. Find the compensator from measurements y to controls u so as to minimize

$$\lim_{N\to\infty} E\left\{\frac{1}{N}\sum_{n=1}^{n}(x_n^2 + 2u_n^2)\right\}$$

We need first to investigate the conditional distribution of the (scalar) state x_n, given the measurement set Y_n

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$= \{y_1, y_2, \ldots, y_n\}$. Its probability density $P\{x_n | Y_n\}$ satisfies the relation

$$P(x_{n+1}|Y_{n+1}) = P(x_{n+1}|y_{n+1}, Y_n)$$
$$= \text{const} \cdot P(x_{n+1}|Y_n) \cdot P(y_{n+1}|x_{n+1})$$

which is a consequence of Bayes' theorem. Furthermore,

$$P(x_{n+1}|Y_n) = \underset{x_n}{Av}\, P(x_{n+1}, x_n | Y_n)$$
$$= \underset{x_n}{Av} \left\{ P(x_{n+1}|x_n) \cdot P(x_n | Y_n) \right\}$$

and $P(y_{n+1}|x_{n+1})$ and $P(x_{n+1}|x_n)$ are known from the measurement and state update equations.

Neglecting ϵ^2, the steady-state solution to this recurrence formula for $P\{x_n | Y_n\}$ is found (see Appendix A) to be

$$P\{x_n | Y_n\} = \text{const} \cdot (1 + \epsilon\phi \tilde{x}_n^3) \exp\left\{ -\tfrac{1}{2}\left[1 + \epsilon\pi_n^{(1)}\right] \right\} \qquad (1)$$

where $\tilde{x}_n$ denotes $x_n - \hat{x}_n$, $\hat{x}_n$ is the maximum-likelihood estimate of x_n given Y_n, and where

$$\phi = (1/7)F_1 - (4/7)H_1 \qquad (2)$$

$$\pi_{n+1}^{(1)} = (1/4)\pi_n^{(1)} + \big((5/7)H_1 - (3/7)F_1\big) y_{n+1}$$
$$+ \big((9/7)H_1 + (3/7)F_1\big) u_n + \big((9/7)H_1 - (4/7)F_1\big) \hat{x}_n \qquad (3a)$$

$$\hat{x}_{n+1} = (1/2)(\hat{x}_n + u_n + y_{n+1}) + \epsilon\Big\{(1/8)\pi_n^{(1)}(\hat{x}_n + u_n - y_{n+1})$$
$$+ a_{11}y_{n+1}^2 + a_{12}y_{n+1}\hat{x}_n + a_{13}y_{n+1}u_n + a_{22}\hat{x}_n^2$$
$$+ a_{23}\hat{x}_n u_n + a_{33}u_n^2 + a_0\Big\} \qquad (3b)$$

in which

$$a_{11} = (1/14)H_1 + (3/28)F_1, \qquad a_{12} = (2/7)F_1 - (1/7)H_1$$
$$a_{13} = -(3/14)F_1 - (1/7)H_1 \qquad (4a)$$
$$a_{22} = (3/28)F_1 - (3/7)H_1, \quad a_{23} = -(2/7)F_1 - (6/7)H_1 \qquad (4b)$$
$$a_{33} = a_{22}, \qquad a_0 = -(1/7)F_1 - (3/7)H_1 \qquad (4c)$$

This should be compared with the simple conditional density $P(x_n | Y_n) = \text{const} \cdot \exp(-\tfrac{1}{2}\tilde{x}_n^2)$ associated with the steady-state Kalman filter for $\epsilon = 0$. Note that $-\epsilon\pi_n^{(1)}$ (the ϵ correction to the variance of $\tilde{x}_n$) is, like $\tilde{x}_n$, a random variable. Note also that the conditional mean of $\tilde{x}_n$ is not zero but $3\epsilon\phi$.

To find the optimal compensator we now introduce

$$J_n(\hat{x}_n, \pi_n^{(1)}) = \min_{u_n, \ldots, u_N} E\left\{ \sum_{m=n}^{N} (x_m^2 + 2u_m^2) | Y_n \right\}$$

in which u_{n+k} is to be a function of Y_{n+k}. Hence, the recurrence relation

$$J_n(\hat{x}_n, \pi_n^{(1)}) = \min_{u_n} E\left\{ x_n^2 + 2u_n^2 + J_{n+1}(\hat{x}_{n+1}, \pi_{n+1}^{(1)}) | Y_n \right\}$$
$$= \hat{x}_n^2 + (1 - \epsilon\pi_n^{(1)}) + \min_{u_n} \left[2u_n^2 + E\{J_{n+1}(\hat{x}_{n+1}, \pi_{n+1}^{(1)}) | Y_n\} \right]$$

Still neglecting ϵ^2, the steady-state solution $J_n(\hat{x}, \pi^{(1)})$, appropriate for large N, is found to be (see Appendix B)

$$J_n(\hat{x}, \pi^{(1)}) = 2\hat{x}^2 + 3(N - n) + \epsilon(K\pi^{(1)} + L\hat{x} + M\hat{x}^3) \qquad (5)$$

where

$$K = -(10/3), \quad L = (620/21)F_1 - (8/3)H_1, \quad M = (16/7)F_1$$

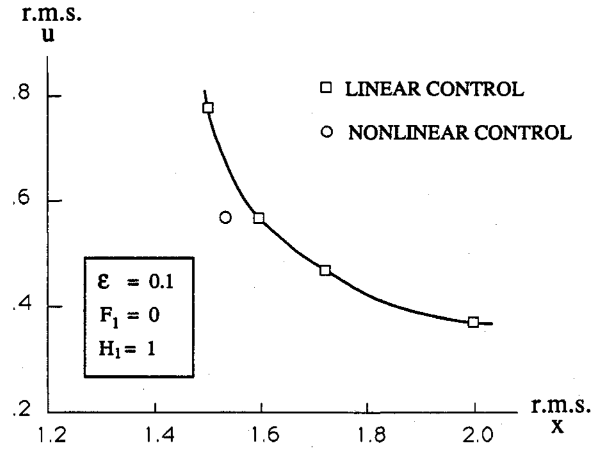

Fig. 1 Summary of numerical results.

and the minimizing control is

$$u_n = -\tfrac{1}{2}\hat{x}_n + \epsilon(c_0 + c_2\hat{x}_n^2) \qquad (6)$$

where

$$c_0 = -(106/21)F_1 + (2/3)H_1, \qquad c_2 = -(5/7)F_1$$

The optimal compensator, then, is given by Eq. (6), in which $\hat{x}_n$ is obtained from measurements by Eqs. (3) and (4), all truncated, of course, at the first power of ϵ.

Numerical Results

Numerical simulation of the controlled system, for various values of ϵF_1 and ϵH_1, has been carried out for 10,000 steps starting from $x_o = u_o = \hat{x}_o = \pi_0^{(1)} = 0$, and the resulting root mean squares of both x and u have been compared with those from simulations of the same nonlinear system controlled by the linear compensator

$$u = -C\hat{x}_n, \qquad \hat{x}_{n+1} = \tfrac{1}{2}(\hat{x}_n + u_n + y_{n+1})$$

for various values of C between 0 and 1, corresponding to varying the weight 2 on u_n^2 in the cost function.

This is shown in Fig. 1 for $\epsilon = 0.1$, $F_1 = 1$, and $H_1 = 0$, where the advantage of the nonlinear compensator is clear if not dramatic. For larger ϵ, of course, we can expect a greater advantage. Indeed, for $\epsilon \geq 0.15$, the linear compensator failed to stabilize the system for 1000 steps. The same occurred for $(F_1, H_1) = (1,0)$, $(1,1)$, and $(1, -1)$ when $\epsilon \geq 0.1$. In all of these cases, the nonlinear compensator was satisfactory for ϵ up to 5 times larger. The eventual instability of the nonlinear compensator must be due to truncation at the first power of ϵ. Inclusion of ϵ^2 terms, a very tedious exercise, would include an addition to the control proportional to $-\epsilon^2\hat{x}^3$ as well as terms involving parameters of the conditional state distribution other than $\hat{x}$.

Concluding Remarks

The present simple example turns out to be a difficult one because in the scalar case a quadratic nonlinearity is just a square term that always drives to instability. The linear compensator goes out of control due to occasional large random fluctuations, unless the nonlinearity is quite small. The nonlinear compensator is satisfactory for considerably larger nonlinearity. A still larger operational domain would be anticipated if the present analysis were carried up to second order. This would enable one to attach uncertainties to quadratic coefficients, and at the same time it would decrease the truncation error in the compensator.

Appendix A: Estimator Derivation

First, neglecting ϵ altogether, $P(x_n \mid Y_n)$ should have the following simple form:

$$P(x_n \mid Y_n) = \text{const} \cdot \exp\left[-\tfrac{1}{2}\pi_n(x_n - \hat{x}_n)^2\right] \qquad \text{(A1)}$$

where π_n is the reciprocal of the variance of this Gaussian distribution.

Substituting into the recurrence formula

$$P(x_{n+1} \mid Y_{n+1}) = \text{const} \cdot P(y_{n+1} \mid x_{n+1}) \cdot \underset{x_n}{\text{Av}}\{P(x_{n+1} \mid x_n) \times P(x_n \mid Y_n)\}$$

we find

$$\exp\left[-\tfrac{1}{2}\pi_{n+1}(x_{n+1} - \hat{x}_{n+1})^2\right] = \text{const} \exp\left[-\tfrac{1}{4}(y_{n+1} - x_{n+1})^2\right] \times \int \exp\left[-\tfrac{1}{2}(x_{n+1} - x_n - u_n)^2 - \tfrac{1}{2}\pi_n(x_n - \hat{x}_n)^2\right] \mathrm{d}x_n \qquad \text{(A2)}$$

The integration over x_n is straightforward since the exponent in the integrand may be written as

$$-\tfrac{1}{2}(1+\pi_n)x_n^2 + x_n(x_{n+1} - u_n + \pi_n\hat{x}_n) - \tfrac{1}{2}(x_{n+1} - u_n)^2 - \tfrac{1}{2}\pi_n\hat{x}_n^2 = -\tfrac{1}{2}(1+\pi_n) \times \left\{x_n - \frac{x_{n+1} - u_n + \pi_n\hat{x}_n}{1+\pi_n}\right\}^2 + \frac{(x_{n+1} - u_n + \pi_n\hat{x}_n)^2}{2(1+\pi_n)} - \tfrac{1}{2}(x_{n+1} - u_n)^2 - \tfrac{1}{2}\pi_n\hat{x}_n^2 \qquad \text{(A3)}$$

The exponential of the first term integrates to an expression, $\sqrt{2\pi/(1+\pi_n)}$, independent of x_{n+1}. The logarithm of the right side of Eq. (A2) thus becomes

$$\left[-\frac{3}{4} + \frac{1}{2(1+\pi_n)}\right]x_{n+1}^2 + \left[\frac{1}{2}y_{n+1} + \frac{\pi_n\hat{x}_n - u_n}{1+\pi_n} + u_n\right]x_{n+1} + \text{terms independent of } x_{n+1}$$

Comparison with the logarithm of the left side of Eq. (A2) now shows that

$$\pi_{n+1} = 3/2 - 1/(1+\pi_n)$$

and that in the steady-state

$$\hat{x}_{n+1} = \tfrac{1}{2}(\hat{x}_n + u_n + y_{n+1}) \qquad \text{(A4)}$$

This, of course, is the usual steady-state Kalman filter:

$$\hat{x}_{n+1} = \hat{x}_n + u_n + K(y_{n+1} - \hat{x}_n - u_n)$$

with $K = \tfrac{1}{2}$.

Restoring the first power of ϵ, the right side of Eq. (A2) will include ϵ terms, say ϵZ, given by

$$\epsilon Z = \epsilon\{F_1 x_n^2(x_{n+1} - x_n - u_n) + \tfrac{1}{2}H_1 x_{n+1}^2(y_{n+1} - x_{n+1})\} \qquad \text{(A5)}$$

in the exponent, and, hence, in the right side itself, as well as the effect of ϵ corrections to the expression (A1) for $P(x_n \mid Y_n)$.

Integration of the first ϵ term over x_n, using of course just the approximation [Eq. (A1)] for $P(x_n \mid Y_n)$ and the steady-state value $\pi = 1$, clearly produces an ϵ term cubic in $\hat{x}_n$ as well as lower powers of $\hat{x}_n$. We may therefore assume, in the steady state, that $P(x_n \mid Y_n)$ has the form

$$P(x_n \mid Y_n) = \text{const} \cdot \exp\left[-\tfrac{1}{2}(1 + \epsilon\pi_n^{(1)})\tilde{x}_n^2\right] \times \{1 + \epsilon\phi_n\tilde{x}_n^3\} + \mathcal{O}(\epsilon^2) \qquad \text{(A6)}$$

where $\tilde{x}_n$ denotes $x_n - \hat{x}_n$, but where the updating of $\hat{x}_n$, as in Eq. (A4), will include ϵ terms to be determined. $(-\epsilon\pi_n^{(1)})$ is a correction to the unit variance of the distribution, and $\pi_n^{(1)}$, like $\hat{x}_n$, will be a random variable whose evolution depends on the measurements. The coefficient ϕ_n in Eq. (A6) will be found to have a steady-state value to be determined.

The modification of Eq. (A2) is now

$$\exp\left[-\tfrac{1}{2}(1 + \epsilon\pi_{n+1}^{(1)})(x_{n+1} - \hat{x}_{n+1})^2\right]\{1 + \epsilon\phi_{n+1}(x_{n+1} - \hat{x}_{n+1})^3\} = \text{const} \cdot \exp\left[-\tfrac{1}{4}(y_{n+1} - x_{n+1})^2\right] \times \int \exp\left[-\tfrac{1}{2}(x_{n+1} - x_n - u_n)^2 - \tfrac{1}{2}(x_n - \hat{x}_n)^2\right] \times \{1 - \tfrac{1}{2}\epsilon\pi_n^{(1)}(x_n - \hat{x}_n)^2 + \epsilon\phi_n(x_n - \hat{x}_n)^3 + \epsilon Z\}\, \mathrm{d}x_n \qquad \text{(A7)}$$

The integration over x_n of the ϵ terms on the right side of Eq. (A7) is obtainable because of Eq. (A3) with $\pi_n = 1$, simply by replacing

x_n by $\tfrac{1}{2}(x_{n+1} - u_n + \hat{x}_n)$

x_n^2 by $\tfrac{1}{4}(x_{n+1} - u_n + \hat{x}_n)^2 + \tfrac{1}{2}$

x_n^3 by $\tfrac{1}{8}(x_{n+1} - u_n + \hat{x}_n)^3 + \tfrac{3}{4}(x_{n+1} - u_n + \hat{x}_n)$

The logarithm, say L^*, of the right side of Eq. (A7), thus becomes

$$\begin{aligned} L^* = {} & -\tfrac{1}{2}x_{n+1}^2 + \tfrac{1}{2}(\hat{x}_n + u_n + y_{n+1})x_{n+1} \\ & + \epsilon(\tfrac{1}{8}\phi_n + \tfrac{1}{8}F_1 - \tfrac{1}{2}H_1)x_{n+1}^3 \\ & + \epsilon\{-\tfrac{1}{8}\pi_n^{(1)} - \tfrac{3}{8}\phi_n(\hat{x}_n + u_n) \\ & + F_1(\tfrac{1}{8}\hat{x}_n - \tfrac{3}{8}u_n) + \tfrac{1}{2}H_1 y_{n+1}\}x_{n+1}^2 \\ & + \epsilon\{\tfrac{1}{4}\pi_n^{(1)}(\hat{x}_n + u_n) + \tfrac{3}{8}\phi_n(\hat{x}_n + u_n)^2 + \tfrac{3}{4}\phi_n \\ & + F_1(-\tfrac{1}{4} - \tfrac{1}{8}\hat{x}_n^2 - \tfrac{1}{4}\hat{x}_n u_n + \tfrac{3}{8}u_n^2)\}x_{n+1} \\ & + \text{terms independent of } x_{n+1} + \mathcal{O}(\epsilon^2) \end{aligned} \qquad \text{(A8)}$$

Comparing the coefficient of x_{n+1}^3 in L^* with that of the logarithm of the left side of Eq. (A7), we obtain

$$\phi_{n+1} = \tfrac{1}{8}\phi_n + \tfrac{1}{8}F_1 - \tfrac{1}{2}H_1$$

whose steady-state solution is Eq. (2).

Next, comparing the coefficients of x_{n+1}^2, using just the zeroth approximation [Eq. (A4)] to $\hat{x}_{n+1}$ in the expression $-3\epsilon\phi_{n+1}\hat{x}_{n+1}$ on the left side, and replacing ϕ_n, ϕ_{n+1} by their steady-state value Eq. (2), we obtain $\pi_{n+1}^{(1)}$ as in Eq. (3a).

Finally, comparing the coefficients of x_{n+1}, again using Eq. (A4) in the left side expressions $3\epsilon\phi_{n+1}\hat{x}_{n+1}^2$ and $\epsilon\pi_{n+1}^{(1)}\hat{x}_{n+1}$, and substituting from Eq. (2) for ϕ_n, ϕ_{n+1}, and from Eq. (3a) for $\pi_{n+1}^{(1)}$, we obtain Eqs. (3b) and (4).

Appendix B: Compensator Derivation

Again, we start by neglecting ϵ altogether. J_n, now a function of $\hat{x}_n$ only, satisfies the recurrence relation:

$$J_n(\hat{x}_n) = \hat{x}_n^2 + 1 + \min_{u_n}\left[2u_n^2 + E\{J_n(\hat{x}_{n+1}) \mid Y_n\}\right] \qquad \text{(B1)}$$

whose solution obviously has the form

$$J_n(\hat{x}_n) = S_n\hat{x}_n^2 + \Sigma_n \qquad \text{(B2)}$$

Noting that

$$\hat{x}_{n+1} = \hat{x}_n + u_n + \tfrac{1}{2}(\tilde{x}_n + w_n + w'_{n+1})$$

and that $\tilde{x}_n$, w_n, w'_{n+1} have variance 1, 1, and 2, respectively,

$$E(\hat{x}_{n+1}^2 \mid Y_n) = (\hat{x}_n + u_n)^2 + 1$$

and, hence, the minimizing u_n in Eq. (B1) satisfies

$$4u_n + 2S_{n+1}(\hat{x}_n + u_n) = 0 \tag{B3}$$

and, hence, from Eq. (B1):

$$S_n\hat{x}_n^2 + \Sigma_n = \hat{x}_n^2 + 1 + \frac{2S_{n+1}^2\hat{x}_n^2}{(2+S_{n+1})^2} + S_{n+1}\left\{\frac{4\hat{x}_n^2}{(2+S_{n+1})^2} + 1\right\} + \Sigma_{n+1} \tag{B4}$$

This implies that

$$S_n = \frac{2S_{n+1}}{2+S_{n+1}} + 1 \tag{B5a}$$

$$\Sigma_n = \Sigma_{n+1} + (1 + S_{n+1}) \tag{B5b}$$

Backward evaluation of Eqs. (B5) shows that $S_n \to 2$ so that $u_n \to -\frac{1}{2}\hat{x}_n$, while Σ_n accumulates:

$$\Sigma_n \cong 3(N-n) \quad \text{for} \quad N \gg n$$

Restoring the first power of ϵ and substiuting for y_{n+1} in terms of x_{n+1} and w'_{n+1}, for x_{n+1} in terms of x_n, u_n, and w_n, and replacing x_n by $\hat{x}_n + \tilde{x}_n$, we obtain

$$\begin{aligned}\hat{x}_{n+1} = \hat{x}_n + u_n &+ \left[\tfrac{1}{2} - (\epsilon/8)\pi_n^{(1)}\right](\tilde{x}_n + w_n + w'_{n+1}) \\ &+ \epsilon F_1\left\{\hat{x}_n^2 + \hat{x}_n\tilde{x}_n + \tfrac{1}{2}\hat{x}_n(\tilde{x}_n + w_n + w'_{n+1})\right. \\ &\left. + \tfrac{1}{2}\tilde{x}_n^2 + (3/28)\,(\tilde{x}_n + w_n + w'_{n+1})^2 - (1/7)\right\} \\ &+ \epsilon H_1\left\{(\hat{x}_n + u_n)(\tilde{x}_n + w_n) + \tfrac{1}{2}(\tilde{x}_n + w_n)^2\right. \\ &\left. + (1/14)(\tilde{x}_n + w_n + w'_{n+1})^2 - (3/7)\right\}\end{aligned} \tag{B6}$$

Recalling that in the steady state, neglecting ϵ^2:

$$E(\tilde{x}_n^2 \mid Y_n) = 1 - \epsilon\pi_n^{(1)}$$

$$E(\tilde{x}_n \mid Y_n) = 3\epsilon\phi$$

where ϕ is given by Eq. (2), we find

$$\begin{aligned}E(\hat{x}_{n+1}^2 \mid Y_n) &= (\hat{x}_n + u_n)^2 + 1 - (3/4)\epsilon\pi_n^{(1)} \\ &+ \epsilon F_1\left[2(\hat{x}_n^2 + 1)(\hat{x}_n + u_n) + 3\hat{x}_n\right] + 2\epsilon H_1(\hat{x}_n + u_n)\end{aligned} \tag{B7}$$

This suggests that Eq. (B2), for $N \gg n$, should be replaced by

$$J_n(\hat{x}_n, \pi_n^{(1)}) = 2\hat{x}_n^2 + 3(N-n) + \epsilon(K_n\pi_n^{(1)} + L_n\hat{x}_n + M_n\hat{x}_n^3) \tag{B8}$$

Now, neglecting ϵ, $E(\pi_{n+1}^{(1)} \mid Y_n)$ is found from (3a) to be given by

$$E(\pi_{n+1}^{(1)} \mid Y_n) = \tfrac{1}{4}\pi_n^{(1)} - F_1\hat{x}_n + 2H_1(\hat{x}_n + u_n) \tag{B9}$$

whereas from Eq. (B6):

$$E(\hat{x}_{n+1} \mid Y_n) = \hat{x}_n + u_n + \mathcal{O}(\epsilon) \tag{B10a}$$

$$E(\hat{x}_{n+1}^3 \mid Y_n) = (\hat{x}_n + u_n)^3 + 3(\hat{x}_n + u_n) + \mathcal{O}(\epsilon) \tag{B10b}$$

Next, $E\{J_{n+1}(\hat{x}_{n+1}, \pi_{n+1}^{(1)} \mid Y_n\}$ is obtained from Eqs. (B8), (B7), (B9), and (B10), and in the recurrence relation

$$J_n(\hat{x}_n, \pi_n^{(1)}) = \hat{x}_n^2 + (1 - \epsilon\pi_n^{(1)}) + \min_{u_n}\left[2u_n^2 + E\{J_{n+1}(\hat{x}_{n+1}, \pi_{n+1}^{(1)}) \mid Y_n\}\right] \tag{B11}$$

the minimizing u_n differs from the previous $-\frac{1}{2}\hat{x}_n$ only by terms of order ϵ. Substitution of $-\frac{1}{2}\hat{x}_n$ for u_n in Eq. (B11) thus introduces an error only of order ϵ^2, which we neglect. Making this substitution and equating coefficients of $\pi_n^{(1)}$, $\hat{x}_n^3$, and $\hat{x}_n$, we obtain

$$K_n = \tfrac{1}{4}K_{n+1} - (5/2) \tag{B12a}$$

$$M_n = \tfrac{1}{8}M_{n+1} + 2F_1 \tag{B12b}$$

$$L_n = \tfrac{1}{2}L_{n+1} + K_{n+1}(H_1 - F_1) + (3/2)M_{n+1} + 2H_1 + 8F_1 \tag{B12c}$$

Backward iteration of Eqs. (B12) shows that K_n, M_n, and L_n approach the limiting values given in Eq. (5).

Finally, the ϵ correction to u_n is obtainable by equating to zero the partial derivative of Eq. (B12) with respect to u_n, the ϵ terms being evaluated at

$$u_n = -\tfrac{1}{2}x_n$$

Thus,

$$4u_n + 4(\hat{x}_n + u_n) + 4\epsilon F_1(\hat{x}_n^2 + 1) + 4\epsilon H_1 + 2\epsilon KH_1 + \epsilon L + \epsilon M(\tfrac{3}{4}\hat{x}_n^2 + 3) = 0 \tag{B13}$$

The resulting u_n is given by Eq. (6).

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Derivation of the Relative Quaternion Differential Equation

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Introduction

RECENTLY the error analysis of a strapdown inertial navigation system using unit quaternions has been presented in the local vertical coordinates.[1] Though the differential equation for the relative quaternion between body-fixed coordinates and local vertical coordinates is given in Eq. (1) of Ref. 1, a derivation from fundamentals is not provided there. For a complete understanding of the significance of this model with reference to the inertial rate of the body-fixed coordinate system C_{xyz} and the inertial rate of the local vertical coordinate system, it is essential to derive the equations from definitions.

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